

Yang-Mills Theory in 2+1 Dimensions: Coupling of Matter Fields and String-breaking Effects

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Abstract

We explore further the Hamiltonian formulation of Yang-Mills theory in 2+1 dimensions in terms of gauge-invariant matrix variables. Coupling to scalar matter fields is discussed in terms of gauge-invariant fields. We analyze how the screening of adjoint (and other screenable) representations can arise in this formalism. A Schrödinger equation is then derived for the gluelump states which are the daughter states when an adjoint string breaks. A variational solution of this Schrödinger equation leads to an analytic estimate of the string-breaking energy which is within 8.8% of the latest lattice estimates.

1 Introduction

Yang-Mills gauge theories in two spatial dimensions, especially their nonperturbative properties, are interesting for many reasons. First of all, they can be a model for the more realistic, but also more complicated, $(3 + 1)$ -dimensional theories. They have nontrivial dynamical content and propagating degrees of freedom in contrast to Yang-Mills theories in $1 + 1$ dimensions, yet they are more amenable to mathematical analysis with some of the more recent techniques than their $(3 + 1)$ -dimensional counterparts. Beyond being a testing ground for nonperturbative techniques, they are also of direct relevance in, at least, one physical context, namely, Chromodynamics at high temperatures. (Recall that in an imaginary-time formalism, all Matsubara modes are suppressed at high temperatures, except for the zero modes of bosonic fields, leading to a dimensionally reduced theory [1].) Yang-Mills theories in three (or $2 + 1$) dimensions are also of interest from the point of view of gauge-gravity duality, particularly in the context of the recent proposal for string and gravity duals of the maximally supersymmetric Yang-Mills theory in three dimensions [2]. Nonperturbative techniques for strong coupling analyses can also help elucidate some of the details of the gauge-gravity duality.

A few years ago, a Hamiltonian analysis of pure Yang-Mills theory in $2 + 1$ dimensions was developed based on a gauge-invariant matrix parametrization of the gauge potentials [3, 4]. This led to the calculation of a wave functional for the vacuum state of the theory [5]. (For reviews, and other approaches and related papers, see [6, 7, 8].) The vacuum expectation value of the Wilson loop operator could then be evaluated using this wave functional and gave the result

$$\langle W_R(C) \rangle = \exp[-\sigma_R \mathcal{A}_C] \quad (1)$$

where R refers to the representation for the Wilson loop, and \mathcal{A}_C is the area of the loop C . The string tension σ_R was given as

$$\sigma_R = e^4 (c_{ACR}/4\pi) \quad (2)$$

where e is the coupling constant, and c_R , c_A denote the quadratic Casimir values for the representation R and for the adjoint representation, respectively. The area law (1) is consistent with confinement of charged representations. The calculated value of the string tension is in very good agreement with lattice estimates [9, 10]. More recently, motivated

by our Hamiltonian analysis, there has been an attempt to estimate glueball masses [11]. We should note however that the vacuum wave function used in [11], which is somewhat conjectural, agrees with ours, derived in [5], only in the high and low energy limits.¹ An interesting variant of the Hamiltonian analysis is an ‘anisotropic’ formulation with two different couplings corresponding to the two spatial directions [8]. The theory is solvable in the extreme anisotropic limit; however, the approach to the isotropic limit is not entirely clear. In any case, from the results quoted earlier, it is clear that it is worth pursuing and elaborating on our Hamiltonian approach.

The analysis was done in what may be characterized as a continuum strong coupling analysis. (This is different from the strong coupling analysis on the lattice.) For Yang-Mills theory in two spatial dimensions, e^2 has the dimensions of mass and the expansion parameter is e^2/k or k/e^2 , where k is a typical momentum scale. Modes of momenta much smaller than e^2 have to be treated nonperturbatively, and can be analyzed in a k/e^2 expansion, while modes of momenta much larger than e^2 can be treated perturbatively in an e^2/k expansion. Continuum strong coupling refers to an expansion treating modes of small momenta as the dominant contributions to any physical quantity. The nature of the expansion brings up many questions immediately. For example, the area law (1) and the string tension (2) are obtained for any representation, including representations of zero N -ality (say, for the gauge group $SU(N)$). But these representations can be screened. Interpreting the Wilson loop as the propagation of two heavy external charges, we see that a gluon can bind to the external charge producing a color singlet, if the external charges are in representations of zero N -ality. (The color singlet state produced by binding gluons to the external charge is generally referred to as a gluelump.) Thus, when the separation of the external charges becomes large, the energy in the string of gauge fields connecting these charges becomes large and it is energetically favorable to pair-produce gluons which bind to the external charges, neutralizing their color charge. There is no potential between the gluelumps, since they are color-neutral, and so we do not expect an area law for the Wilson loop for these representations. The process of the state of the two heavy charges connected by the string making a transition to the state of two gluelumps is called string-breaking [12, 13]. In our previous analysis, we did not see string-breaking. There is no inconsistency, since we were

¹ As we emphasize in the Appendix, in carrying out the regularization of the Hamiltonian and the “local” operators on which it acts, the order in which the regularization parameter of the Hamiltonian and the point-splitting for the operator are taken to zero is important in preserving important symmetries such as Lorentz invariance. We expect that this is one of the reasons for the discrepancies between the two wave functionals.

using the continuum strong coupling expansion, and string-breaking would not be seen to the order we calculated, but we need to understand how screening can arise as we improve on this approximation.

Closely related to this is the question of whether the formula (2) for the string tension can be exact in any sense. Are there corrections to the formula even for representations of nonzero N -ality? There are arguments in the literature, based on the $1/N$ -expansion, that the ratios of string tensions for different representations should deviate from the ratios of the Casimir invariants [14]. The breaking of the string for a screenable representation is a transition from a single color-singlet operator to two color singlet operators. As such, the transition matrix element is suppressed at large N . So, one possibility is that the formula for the string tension, for the non-screenable representations, is exact in the large N limit. This question was numerically analyzed in detail recently by Teper and Bringoltz [10]. They find that, while the formula (2) for string tension differs from the lattice result by only about 0.88% as $N \rightarrow \infty$, the deviations are still statistically significant. It is therefore important to understand and calculate possible corrections. The approximation of the wave functional for modes of small momenta has a simple form, a Gaussian function in the appropriate variables; this simple form was used in evaluating the vacuum expectation value (1). While the closeness of the agreement between the lattice results and the values given by (2) shows that the Gaussian approximation is a very good first step, the wave functional has non-Gaussian terms and, therefore, it has the potential for explaining corrections to the formula for the string tension. This is an issue that deserves further analysis.

In this paper, we will address the problem of screening of representations of zero N -ality, deferring the issue of corrections to the string tension to a later publication. As mentioned earlier, screening is due to the formation of color singlet gluelump states, where gluons which are pair-created from the vacuum bind to the external charges. Here we will argue for the possibility of such a bound state from the higher non-Gaussian terms of the wave functional. We shall then construct the gluelump state, treating the external charge as a heavy scalar in the adjoint (or some other screenable) representation. A Schrödinger-type eigenvalue equation will be obtained, within certain approximations explained in the text. We then carry out a variational estimate of the gluelump energy. This gives us the estimate for the string-breaking energy which we compare to the value obtained from lattice simulations [13].

For carrying out this analysis, we need to develop a gauge-invariant Hamiltonian approach for scalar fields coupled to the Yang-Mills fields. This will be the subject of the next section. This formalism is also the first step in extending the continuum strong coupling analysis of the gauge theory to the case with matter fields added, and as such, it is of interest beyond understanding the physics of screening. For instance, it can lead to an analysis of the supersymmetric theory, once we have further extended this to include fermion fields. We may note that there has been a recent proposal about the gravity dual for $(2+1)$ -dimensional Yang-Mills theory with sixteen supercharges [2]. The gauge theory in question can be regarded as the dimensional reduction of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory from $R \times S^3$ to $R \times S^2$. A prediction for the leading strong coupling contribution to the masses of the operators built out of the scalars fields has also been computed from the string theory side [2]. Since the gauge-invariant framework that we shall develop in the paper naturally leads to a strong coupling expansion for the gauge theory in the continuum, there is the potential for obtaining similar results directly in the strong coupling regime of the gauge theory and for the possible confirmation of some details of the gauge-gravity duality.

We also note that three-dimensional Yang-Mills theory with matter fields in the adjoint representation naturally arises at finite temperatures in an effective theory of deconfinement [15]. The “matter fields” in this case are the Polyakov loops around the imaginary time direction. They take values in the gauge group; nevertheless some of our analysis can be adapted to analyze that theory. We consider this paper again as a prelude to such an analysis.

In section 2, we construct the Hamiltonian for the $(2+1)$ -dimensional Yang-Mills theory coupled to adjoint scalars in terms of gauge-invariant variables. In section 3, we give qualitative arguments for how screening could arise in the Hamiltonian approach. In section 4, we then use the gauge-invariant framework outlined in section 2 to work out the action of the Hamiltonian for gluelump states of very short spatial extension, appropriate to an extreme strong coupling approximation. This gives a lower bound on the string breaking energy. The extension of this to gluelump states of finite spatial extent is given in section 5 where a Schrödinger-type eigenvalue equation is obtained. In section 6, based on this eigenvalue equation, we obtain an estimate of the string-breaking energy and compare it with available lattice data. There is a short summary/discussion section and the paper

concludes with an appendix with some comments on how regularization has to be carried out for various terms in the Hamiltonian.

2 Scalars coupled to the YM field

In this section, we shall set up the essentials of coupling a real scalar field to Yang-Mills theory in two spatial dimensions. We will consider an $SU(N)$ gauge theory and the scalar field will be in the adjoint representation of $SU(N)$, although various formulae can be easily generalized to any representation.

The action for the theory is

$$S = -\frac{1}{2e^2} \int d^3x \operatorname{Tr}(F^2) + \frac{1}{2} \int d^3x \operatorname{Tr} \left[(D_0\phi)^2 - (\vec{D}\phi)^2 - M^2\phi^2 \right] \quad (3)$$

The corresponding Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \int d^2x \left[\frac{(E^2 + B^2)}{2e^2} + \frac{\dot{\phi}^2 + (\vec{D}\phi)^2 + M^2\phi^2}{2} \right] \\ &= -\frac{1}{2} \int \left[e^2 \frac{\delta^2}{\delta A^a \delta \bar{A}^a} + \frac{\delta^2}{\delta \phi^a \delta \phi^a} \right] + \int \left[\frac{B^a B^a}{2e^2} + \frac{M^2 \phi^a \phi^a}{2} + 2(D\phi)^a (\bar{D}\phi)^a \right] \end{aligned} \quad (4)$$

In writing this expression, we have made the gauge choice $A_0 = 0$. $A = \frac{1}{2}(A_1 + iA_2)$, $\bar{A} = \frac{1}{2}(A_1 - iA_2)$ are the two complex spacial components of the gauge potential. D, \bar{D} are the holomorphic, antiholomorphic covariant derivatives

$$D\phi = \partial\phi + [A, \phi], \quad \bar{D}\phi = \bar{\partial}\phi + [\bar{A}, \phi] \quad (5)$$

In our conventions,

$$A = -iA^a t^a, \quad \phi = \phi^a t^a \quad (6)$$

where the $SU(N)$ generators are normalized such that

$$\operatorname{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}, \quad [t^a, t^b] = i f^{abc} t^c \quad (7)$$

We would like now to rewrite the Hamiltonian in terms of gauge-invariant variables following the approach used in [3, 4, 5] for pure Yang-Mills theory. As before, we use the parametrization

$$A = -\partial M M^{-1}, \quad \bar{A} = M^{\dagger-1} \bar{\partial} M^{\dagger} \quad (8)$$

If the gauge transformations take values in the Lie group G , M is a complex matrix taking values in $G^{\mathbb{C}}$, the complexification of G . For the case of $G = SU(N)$, which is what we shall consider in this paper, $M \in SU(N)^{\mathbb{C}} = SL(N, \mathbb{C})$.

Time-independent gauge transformations are realized as left rotations on the matrix M ,

$$M(\vec{x}) \rightarrow g(\vec{x})M(\vec{x}) , \quad g(\vec{x}) \in SU(N) \quad (9)$$

As a result, the theory can be entirely rewritten in terms of the gauge-invariant variables

$$H = M^\dagger M , \quad \chi = M^\dagger \phi M^{\dagger-1} \quad (10)$$

Before we derive the expression of the full Hamiltonian (4) in terms of these gauge-invariant variables, it is useful to first review the case of the pure Yang-Mills theory. The simplification of the Hamiltonian in terms of the variables M, M^\dagger is as follows. Introduce the right-translation operator p_a for M and the left-translation operator \bar{p}_a for M^\dagger by

$$\begin{aligned} [p_a(\vec{x}), M(\vec{y})] &= M(\vec{y})(-it_a)\delta^{(2)}(\vec{x} - \vec{y}) \\ [\bar{p}_a(\vec{x}), M^\dagger(\vec{y})] &= (-it_a)M^\dagger(\vec{y})\delta^{(2)}(\vec{x} - \vec{y}) \end{aligned} \quad (11)$$

It was shown in [4] that in terms of these, the kinetic energy operator T may be written as

$$\begin{aligned} T &= -\frac{e^2}{2} \int \frac{\delta^2}{\delta A^a \delta \bar{A}^a} = \frac{e^2}{2} \int_{u,v} \Pi_{rs}(\vec{u}, \vec{v}) \bar{p}_r(\vec{u}) p_s(\vec{v}) \\ \Pi_{rs}(\vec{u}, \vec{v}) &= \int_x \bar{\mathcal{G}}_{ar}(\vec{x}, \vec{u}) K_{ab}(\vec{x}) \mathcal{G}_{bs}(\vec{x}, \vec{v}) \end{aligned} \quad (12)$$

where $K_{ab} = 2\text{Tr}(t_a H t_b H^{-1})$ is the adjoint representative of H , and

$$\begin{aligned} \bar{\mathcal{G}}_{ma}(\vec{x}, \vec{y}) &= \frac{1}{\pi(x - y)} \left[\delta_{ma} - e^{-|\vec{x} - \vec{y}|^2/\epsilon} (K(x, \vec{y}) K^{-1}(y, \vec{y}))_{ma} \right] \\ \mathcal{G}_{ma}(\vec{x}, \vec{y}) &= \frac{1}{\pi(\bar{x} - \bar{y})} \left[\delta_{ma} - e^{-|\vec{x} - \vec{y}|^2/\epsilon} (K^{-1}(y, \bar{x}) K(y, \bar{y}))_{ma} \right] \end{aligned} \quad (13)$$

These are the regularized versions of the corresponding Green's functions

$$\bar{G}(\vec{x}, \vec{y}) = \frac{1}{\pi(x - y)} , \quad G(\vec{x}, \vec{y}) = \frac{1}{\pi(\bar{x} - \bar{y})} \quad (14)$$

The parameter controlling the regularization, ϵ , acts as a short-distance cut-off. Expression (12) is to be used on functionals where the point-separation of various factors is much larger than $\sqrt{\epsilon}$. (See Appendix for more details on regularization.)

As discussed in [4] there is an ambiguity in the parametrization (8) : (M, M^\dagger) and $(M\bar{V}(\bar{z}), V(z)M^\dagger)$ give the same gauge potentials A, \bar{A} , where \bar{V}, V are, respectively, antiholomorphic and holomorphic in the complex coordinates $\bar{z} = x_1 + ix_2$ and $z = x_1 - ix_2$. As a result, all physical observables in the theory should satisfy the so-called holomorphic invariance

$$H \rightarrow V(z)H\bar{V}(\bar{z}) \quad (15)$$

The particular choice of regularization used in (13) respects this holomorphic invariance.

When the theory is rewritten in terms of gauge-invariant variables, p_a, \bar{p}_a can be realized as translation operators on $H = M^\dagger M$,

$$\begin{aligned} [p_a(\vec{x}), H(\vec{y})] &= H(\vec{y})(-it_a)\delta^{(2)}(\vec{x} - \vec{y}) \\ [\bar{p}_a(\vec{x}), H(\vec{y})] &= (-it_a)H(\vec{y})\delta^{(2)}(\vec{x} - \vec{y}) \end{aligned} \quad (16)$$

In [3, 4], we have argued that the wave functions can be taken to be functionals of the current $J = (c_A/\pi)\partial H H^{-1}$, where c_A is the quadratic Casimir invariant defined by $c_A\delta^{ab} = f^{amn}f^{bm n}$. The action of the operators p, \bar{p} on the current J is expressed in terms of the commutation relations

$$\begin{aligned} [p_s(\vec{v}), J_a(\vec{z})] &= -i\frac{c_A}{\pi}K_{as}(\vec{z})\partial_z\delta(\vec{z} - \vec{v}) \\ [\bar{p}_r(\vec{u}), J_b(\vec{w})] &= -i(\mathcal{D}_w)_{br}\delta(\vec{w} - \vec{u}) \end{aligned} \quad (17)$$

where $\mathcal{D}_w{}_{ab} = \frac{c_A}{\pi}\partial_w\delta_{ab} + if_{abc}J_c(\vec{w})$. Using (12) and (17), the action of T on wavefunctions of the form $\Psi(J)$ can be expressed as

$$T_{YM}\Psi(J) = \frac{e^2 c_A}{2\pi} \left[\int_z \omega^a(\vec{z}) \frac{\delta}{\delta J^a(\vec{z})} + \int_{z,w} \Omega^{ab}(\vec{z}, \vec{w}) \frac{\delta}{\delta J^a(\vec{z})} \frac{\delta}{\delta J^b(\vec{w})} \right] \Psi(J) \quad (18)$$

where

$$\begin{aligned} \Lambda_{ra}(\vec{w}, \vec{z}) &= -[\partial_z \Pi_{rs}(\vec{w}, \vec{z})] K_{sa}^{-1}(\vec{z}) \\ \omega_a(\vec{z}) &= if_{arm}\Lambda_{rm}(\vec{u}, \vec{z})|_{\vec{u} \rightarrow \vec{z}} \\ \Omega_{ab}(\vec{z}, \vec{w}) &= \mathcal{D}_w{}_{br}\Lambda_{ra}(\vec{w}, \vec{z}) \end{aligned} \quad (19)$$

The above expressions have been computed in [4] where we found that for small ϵ , T can be further simplified as

$$T_{YM}\Psi(J) = m \left[\int J_a(\vec{z}) \frac{\delta}{\delta J_a(\vec{z})} + \int (\mathcal{D}_w \bar{G}(\vec{z}, \vec{w}))_{ab} \frac{\delta}{\delta J_a(\vec{w})} \frac{\delta}{\delta J_b(\vec{z})} \right] \Psi(J) + \mathcal{O}(\epsilon) \quad (20)$$

where $m = e^2 c_A / 2\pi$.

As for the potential energy term, it is trivially checked that we can write

$$V_{YM} = \int d^2x \frac{B^a B^a}{2e^2} = \frac{\pi}{mc_A} \int d^2x : \bar{\partial} J^a \bar{\partial} J^a : \quad (21)$$

The regularized form of this expression is

$$\begin{aligned} V_{YM} &= \frac{\pi}{mc_A} \left[\int_{x,y} \sigma(\vec{x}, \vec{y}; \lambda) \bar{\partial} J_a(\vec{x}) (K(x, \bar{y}) K^{-1}(y, \bar{y}))_{ab} \bar{\partial} J_b(\vec{y}) - \frac{c_A \dim G}{\pi^2 \lambda^2} \right] \quad (22) \\ \sigma(\vec{x}, \vec{y}; \lambda) &= \frac{e^{-\frac{|\vec{x}-\vec{y}|^2}{\lambda}}}{\pi \lambda} \end{aligned}$$

where $\sigma(\vec{x}, \vec{y}; \lambda)$ is a regularized δ -function, λ is the parameter of regularization, and we should take the limit where $\sqrt{\lambda} \ll 1/e^2$. The operator $U^{ab}(\vec{x}, \vec{y}) = [K(x, \bar{y}) K^{-1}(y, \bar{y})]^{ab}$ in (23) is a "holomorphic" Wilson line connecting the two $\bar{\partial} J$'s at different points. $U^{ab}(\vec{x}, \vec{y})$ is such that the regularized expression for V_{YM} satisfies holomorphic invariance. In considering the application of T on this, and on other expressions in the rest of this paper, the correct procedure is to preserve $\lambda \gg \epsilon$ as we take these quantities to zero. This is what was used in [4, 5]; it is also briefly explained in the appendix.

Since T as defined in (12) is valid only for separations much larger than $\sqrt{\epsilon}$, we need to use $\lambda \gg \epsilon$ to define the potential energy term. Correspondingly, we can define the kinetic term also with a scale λ by the equation [4]

$$\begin{aligned} T(\lambda) &= T + \frac{e^2}{2} \log(2\epsilon/\lambda) \mathcal{Q} \\ \mathcal{Q} &= \epsilon \int \sigma(\vec{u}, \vec{v}; \epsilon) K_{rs}(u, \bar{v}) \left(\bar{p}_r(\vec{u}) - i \bar{\partial} J_r(\vec{u}) \right) p_s(\vec{v}) \quad (23) \end{aligned}$$

The understanding here is that $l \gg \epsilon$; this operator is to be taken to act on functionals of J with a point-separation much larger than $\sqrt{\epsilon}$, with $\epsilon \rightarrow 0$ and $\sqrt{l} \ll 1/e^2$.

We now turn to the case with the scalar field added. Evidently the term $\int \frac{\delta^2}{\delta A \delta A}$ does not change and we can still use the expression (12). The full kinetic energy operator T is now

$$\begin{aligned} T &= -\frac{1}{2} \int e^2 \frac{\delta^2}{\delta A^a \delta \bar{A}^a} - \frac{1}{2} \int \frac{\delta^2}{\delta \phi^a \delta \bar{\phi}^a} \\ &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) \bar{p}_r(\vec{u}) p_s(\vec{v}) - \frac{1}{2} \int \frac{\delta^2}{\delta \chi^a \delta \bar{\chi}^a} \quad (24) \end{aligned}$$

where χ is the gauge-invariant field defined in (10). The action of p , \bar{p} on χ is given by

$$\begin{aligned} [p_a(\vec{x}), \chi_b(\vec{y})] &= 0 \\ [\bar{p}_a(\vec{x}), \chi_b(\vec{y})] &= -f_{abc} \chi_c(\vec{y}) \delta^{(2)}(\vec{x} - \vec{y}) \end{aligned} \quad (25)$$

The gauge-invariant wave function, $\Psi(J, \chi)$, is now a functional of both gauge-invariant variables J and χ . In terms of its action on such functionals T can be written as

$$T = T_{YM} + im \int_{z,w} \Lambda_{cd}(\vec{w}, \vec{z}) f^{abc} \chi^a(\vec{w}) \frac{\delta}{\delta \chi^b(\vec{w})} \frac{\delta}{\delta J^d(\vec{z})} - \frac{1}{2} \int \frac{\delta^2}{\delta \chi^a \delta \chi^a} \quad (26)$$

where T_{YM} and $\Lambda_{cd}(\vec{w}, \vec{z})$ are given in (18), (19).

Similarly the full potential energy term can be expressed in terms of gauge-invariant fields as

$$\begin{aligned} V &= \int \left(\frac{B^a B^a}{2e^2} + 2(D\phi)^a (\bar{D}\phi)^a + \frac{M^2 \phi^a \phi^a}{2} \right) \\ &= V_{YM} + \int \left(\frac{2\pi}{c_A} \bar{\partial} \chi^a (\mathcal{D}\chi)^a + \frac{M^2 \chi^a \chi^a}{2} \right) \end{aligned} \quad (27)$$

For purposes of easy reference, we shall now collect various formulae and write the final form of the Hamiltonian for the Yang-Mills field coupled to adjoint scalars as

$$\begin{aligned} \mathcal{H} &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) \bar{p}_r(\vec{u}) p_s(\vec{v}) - \frac{1}{2} \int \frac{\delta^2}{\delta \chi^a \delta \chi^a} + V_{YM} + \int \left(\frac{2\pi}{c_A} \bar{\partial} \chi^a (\mathcal{D}\chi)^a + \frac{M^2 \chi^a \chi^a}{2} \right) \\ &= m \left[\int J_a(\vec{z}) \frac{\delta}{\delta J_a(\vec{z})} + \int (\mathcal{D}_w \Lambda(\vec{w}, \vec{z}))_{ba} \frac{\delta}{\delta J_a(\vec{z})} \frac{\delta}{\delta J_b(\vec{w})} \right] + \frac{\pi}{mc_A} \int : \bar{\partial} J^a \bar{\partial} J^a : \\ &\quad + im \int_{z,w} \Lambda_{cd}(\vec{w}, \vec{z}) f^{abc} \chi^a(\vec{w}) \frac{\delta}{\delta \chi^b(\vec{w})} \frac{\delta}{\delta J^d(\vec{z})} \\ &\quad - \frac{1}{2} \int \frac{\delta^2}{\delta \chi^a \delta \chi^a} + \int \left(\frac{2\pi}{c_A} \bar{\partial} \chi^a (\mathcal{D}\chi)^a + \frac{M^2 \chi^a \chi^a}{2} \right) \end{aligned} \quad (28)$$

It is evident from our derivation that this formula for the Hamiltonian is trivially generalizable to the case of several scalar fields.

3 Screening: a preliminary qualitative analysis

The next logical step is the application of the gauge-invariant formalism of the previous section to the gluelump state. But before doing so, it is useful to go over some qualitative considerations of how screening of the adjoint and other representations of zero N -ality could

arise in our Hamiltonian approach. As discussed in [5], the computation of the expectation value of the Wilson line in pure Yang-Mills theory is of the form

$$\begin{aligned}\langle W_R(C) \rangle &= \int d\mu(\mathcal{A}/\mathcal{G}_*) e^{-S(H)} W_R(C) \\ &= \int d\mu(H) e^{2c_A S_{wzw}(H)} e^{-S(H)} W_R(C)\end{aligned}\quad (29)$$

where $d\mu(\mathcal{A}/\mathcal{G}_*) = d\mu(H) e^{2c_A S_{wzw}(H)}$ is the gauge-invariant volume element for Yang-Mills fields and $e^{-S(H)}$ is given in terms of the vacuum wave function as $e^{-S(H)} = \Psi_0^* \Psi_0$. $S_{wzw}(H)$ is the Wess-Zumino-Witten action for the hermitian field H and $d\mu(H)$ is the Haar measure for H viewed as an element of $SL(N, \mathbb{C})/SU(N)$. Explicitly,

$$S_{wzw}(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_\mu H H^{-1} \partial_\nu H H^{-1} \partial_\alpha H) \quad (30)$$

The expression for the wave function Ψ_0 shows that $S(H)$ has the form

$$\begin{aligned}S(H) &= \frac{4\pi^2}{e^2 c_A^2} \int \bar{\partial} J_a \left[\frac{1}{(m + \sqrt{m^2 - \nabla^2})} \right] \bar{\partial} J_a \\ &\quad - 2f_{abc} \int f^{(3)}(\vec{x}, \vec{y}, \vec{z}) J_a(\vec{x}) J_b(\vec{y}) J_c(\vec{z}) + \mathcal{O}(J^4)\end{aligned}\quad (31)$$

The function $f^{(3)}(\vec{x}, \vec{y}, \vec{z})$ has been given in [5].

Equation (29) shows that expectation values can be considered as averages in a two-dimensional field theory with the action $S(H)$. In evaluating the string tension and obtaining the formula in terms of the Casimir values of the representations, we have basically used the low-momentum limit of the quadratic terms (terms which are quadratic in the current J) in this functional. These terms correspond to

$$S(H) = \frac{1}{4g^2} \int d^2x F_{ij}^a F_{ij}^a \quad (32)$$

where $g^2 = me^2 = e^4 c_A / 2\pi$. In this approximation, the expectation value reduces to the average calculated using a Euclidean two-dimensional Yang-Mills theory with coupling constant g . There are many ways to evaluate these averages, one simple way is to go back to the original variables, the A 's, and calculate, for example, in a gauge $A_1 = 0$. In this gauge, the commutator term in F_{ij} is absent and we have a Gaussian functional integral. This leads directly to the formula for the string tension.

As mentioned in the introduction, the values we find are in very good agreement with the lattice analysis [9, 10], although the lattice results are accurate enough to show a

statistically significant deviation, even at large N , where the agreement is the best. Also, it has been argued, on the basis of the $1/N$ -expansions, that deviations from the Casimir scaling for string tensions of different representations are possible [14]. Further, we know, on general theoretical grounds, that the Wilson loops in representations of zero N -ality can be screened and will not show the area law behavior for large enough loops. Therefore, we turn to considerations of possible corrections to this result. The corrections can arise from the higher terms involving three or more powers of the current J . The string tension is defined by the limit of large Wilson loops. Therefore we may think of “integrating out” the high momentum modes in the two-dimensional theory defined by $S(H)$ to obtain an effective low-momentum action $S_{eff}(H)$ and then using this to calculate the average of the Wilson loop. It is then easy to see that there are two types of corrections possible. The first set would correspond to a corrected quadratic term, due to terms which modify the “propagator” for the currents. This is equivalent to a modified coupling g_* in the effective two-dimensional Yang-Mills theory. Since this is at the level of the action and we have not yet introduced the Wilson line, there is no dependence on the representation of the Wilson line; it depends only on N and numerical factors. Therefore, these corrections can affect the value of σ_R , but the ratio σ_R/σ_F is unaffected.

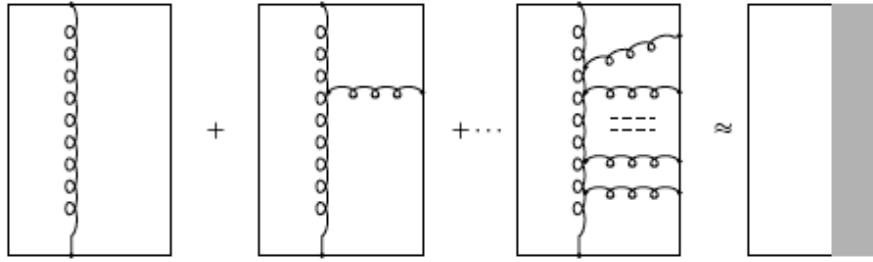
The second set of corrections involve the J^3 and higher vertices with the currents connected to the Wilson line via propagators. These cannot be reduced to propagator corrections and hence we can get representation-dependent modifications. In particular, screening effects have to come from such terms. In principle, all the J^3 and higher vertices can contribute. To elucidate the nature of these terms, consider the contributions from the cubic vertex [5]. Written in terms of the original variables, the cubic term is of the form $\text{Tr}(\bar{D}^{-1}B[B, \bar{D}B])$ (Since we have phrased the calculation of the string tension in terms of the A ’s, we use the same variables for this argument.) Expanding \bar{D}^{-1} in powers of \bar{A} , we see that

$$\bar{D}^{-1} = \bar{\partial}^{-1} - \bar{\partial}^{-1}\bar{A}\bar{\partial}^{-1} + \dots \quad (33)$$

In the $A_1 = 0$ gauge, the cubic term is thus of the form

$$S_{int}^{(3)} \sim [(\bar{\partial}^{-1} - \bar{\partial}^{-1}\bar{A}\bar{\partial}^{-1} + \dots) \partial_1 A_2]^a \partial_1 A_2^b \partial_1 \bar{\partial} A_2^c f^{abc} \quad (34)$$

Because of the overall Euclidean invariance, the Wilson line can be visualized as the propagation of two heavy charges. Consider then the insertion of the interaction term (34). The contractions of the A ’s from the expansion of \bar{D}^{-1} will produce a sequence of terms which is



A set of terms whose sum can be viewed as the propagation kernel of a bound state
(shown as thick gray line)

effectively like the iteration of the kernel in a Bethe-Salpeter equation for a bound state. In other words, we get terms which can be interpreted as the propagation kernel of a possible bound state of the gluons with the external charge of the Wilson line. For the nonscreenable representations, it is not meaningful to think of the bound state in isolation since it is not a color singlet. But for the screenable representations, a color singlet bound state can be formed. The picture which emerges is one where gluons are created from the vacuum and then bind to the external charges to form color singlets. This can lead to breaking of the single string connecting the external charges into two such singlet bound states.

As for the quantitative calculation of these corrections, we know from the lattice data that they are small, except for the screening effects. It is, of course, important to calculate the corrections and provide an analytic justification for why they are small. The calculation of these corrections and the analytic justification for why they are small will be reported in another paper. Here we will pursue the analysis of string breaking for screenable representations. There are two related issues for the analysis of this question. There is a transition matrix element between the state of the single string connecting the external charges and the two singlet bound states. This matrix element will describe the process of the string-breaking, and, from general arguments, it should be suppressed by powers of N , in a large N expansion. The calculation of this matrix element is quite involved and we will not carry this out here. However, since the energy at which the string breaks corresponds to twice the energy of the singlet bound state, we can calculate this string-breaking energy by calculating the energy of the singlet bound state of a gluon with the external heavy charge (whose propagation is one side of the Wilson line). This bound state has generally been

referred to as the gluelump. In our language, it corresponds to a wave function of the form

$$\begin{aligned}\Psi_G &= \int d^2x d^2y f(\vec{x}, \vec{y}) \bar{\partial} J^a(\vec{x}) [K(x, \vec{y}) K^{-1}(y, \vec{y})]^{ab} \chi^b(\vec{y}) \Psi_0 \\ &\equiv \mathcal{O}_G \Psi_0\end{aligned}\tag{35}$$

Here J represents the gauge-invariant gluon which has a mass m . In reference [4], we have discussed a constituent picture for the formation of bound states, such as glueballs, in terms of elementary constituents of mass m corresponding to J . The gluelump, as described by (35), is consistent with that picture. The function $f(\vec{x}, \vec{y})$ is then the wave function of the constituents in the bound state.

We can now move onto a more specific and quantitative estimate of the bound state energy rather than the general qualitative arguments given above. For this, the strategy will be to act with the Hamiltonian on the state (35) and obtain a two-particle Schrödinger-like equation for the function $f(\vec{x}, \vec{y})$ characterizing the bound state. The details of this analysis are given in the next section.

4 The action of the Hamiltonian on Ψ_G

We will first consider the (continuum) strong coupling limit where we use the state (35) with the vacuum wave function $\Psi_0 = \exp(-\frac{1}{2}S(H))$ approximated by

$$\begin{aligned}\Psi_0 &\approx \exp \left[-\frac{V_{YM}}{2m} - \frac{M}{2} \int d^2x \chi^2 \right] \\ &= \exp \left[-\frac{1}{8g^2} \int d^2x F^2 - \frac{M}{2} \int d^2x \chi^2 \right]\end{aligned}\tag{36}$$

We have used the perturbative vacuum for the χ -field. In the large M limit, because $M^2 \gg \bar{\partial}\mathcal{D}$, corrections due to the interaction term in $\bar{\partial}\mathcal{D}$ are suppressed by powers of $1/M$.

With the approximation of Ψ_0 as in (36), the action of the Hamiltonian on Ψ_G can be written as²

$$\begin{aligned}\mathcal{H} \Psi_G &= M \Psi_G + \mathcal{O}_G \mathcal{H}_{YM} \Psi_0 + \Psi_0 (T \mathcal{O}_G) \\ &\quad - \frac{e^2}{4m} \int \Pi_{rs}(\vec{u}, \vec{v}) [\bar{p}_r(\vec{u}), V_{YM}] [p_s(\vec{v}), \mathcal{O}_G] \Psi_0 \\ &\quad - \frac{e^2}{4m} \int \Pi_{rs}(\vec{u}, \vec{v}) [p_s(\vec{v}), V_{YM}] [\bar{p}_r(\vec{u}), \mathcal{O}_G] \Psi_0\end{aligned}\tag{37}$$

²We take the Hamiltonian to be normally ordered for the scalar fields so that there is no zero-point energy contribution in (37).

Since Ψ_0 is the vacuum for the Yang-Mills part, $\mathcal{H}_{YM}\Psi_0 = 0$. For the third term and the last term on the right hand side, it should be kept in mind that \bar{p}_r has nontrivial action on χ as given in (25).

In evaluating $(T\mathcal{O}_G)$, it is useful to compare it with the action of the Yang-Mills kinetic term, which we have evaluated elsewhere [4]. Consider

$$\Psi_2 = \int d^2x d^2y f(\vec{x}, \vec{y}) \bar{\partial} J^a(\vec{x}) [K(x, \vec{y}) K^{-1}(y, \vec{y})]^{ab} \bar{\partial} J^b(\vec{y}) \quad (38)$$

Notice that \mathcal{O}_G is obtained from this by replacing the last $\bar{\partial} J$, namely $\bar{\partial} J^b(\vec{y})$, by $\chi^b(\vec{y})$. The action of T on the state (38) can be written as the sum of four terms as given below.

$$T \Psi_2 = I + II + III + IV$$

$$\begin{aligned} I &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) [\bar{p}_r(\vec{u}), [p_s(\vec{v}), \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y})]] \bar{\partial} J^b(\vec{y}) f(\vec{x}, \vec{y}) \\ II &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) [p_s(\vec{v}), \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y})] [\bar{p}_r(\vec{u}), \bar{\partial} J^b(\vec{y})] f(\vec{x}, \vec{y}) \\ III &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) [\bar{p}_r(\vec{u}), \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y})] [p_s(\vec{v}), \bar{\partial} J^b(\vec{y})] f(\vec{x}, \vec{y}) \\ IV &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y}) [\bar{p}_r(\vec{u}), [p_s(\vec{v}), \bar{\partial} J^b(\vec{y})]] f(\vec{x}, \vec{y}) \end{aligned} \quad (39)$$

where $U^{ab}(\vec{x}, \vec{y}) = [K(x, \vec{y}) K^{-1}(y, \vec{y})]^{ab}$. We will consider the evaluation of these terms for small separations $|\vec{x} - \vec{y}| \ll 1/e^2$, with $|\vec{x} - \vec{y}| \gg \sqrt{\epsilon}$. (This means, of course, that the width of f which controls the average value of $|\vec{x} - \vec{y}|$ is small compared to $1/e^2$.) Term I then leads to a mass m for $\bar{\partial} J$ and the Coulomb potential $\mathcal{V}(\vec{x}, \vec{y})$. Terms II and III contain a normal-ordering term, with the p, \bar{p} acting on the $\bar{\partial} J$'s. The other terms arising from these two are negligible for very small separations; these involve at least one power of $|\vec{x} - \vec{y}|$ and three or more J 's. Finally the last term gives m for the final $\bar{\partial} J(\vec{y})$. Thus we find

$$T \Psi_2 = \int_{x,y} [2m + \mathcal{V}(\vec{x}, \vec{y})] f(\vec{x}, \vec{y}) : \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y}) \bar{\partial} J^b(\vec{y}) : + \mathcal{O}(|\vec{x} - \vec{y}|) \quad (40)$$

The Coulomb potential in (40) is of the form

$$\mathcal{V}(\vec{x}, \vec{y}) = m \text{Ein}(|\vec{x} - \vec{y}|^2/2\epsilon) \approx m \log(|\vec{x} - \vec{y}|^2/2\epsilon) \quad (41)$$

where

$$\text{Ein}(z) = \int_0^z \frac{dt}{t} (1 - e^{-t}) \quad (42)$$

In comparing this with $T\mathcal{O}_G$, we see that we will get terms similar to I and II , but the terms analogous to III and IV will be absent when $\bar{\partial}J(\vec{y})$ is replaced by $\chi(\vec{y})$ because of (25). In particular we find

$$\begin{aligned} I' &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) [\bar{p}_r(\vec{u}), [p_s(\vec{v}), \bar{\partial}J^a(\vec{x})U^{ab}(\vec{x}, \vec{y})]] \chi^b(\vec{y}) f(\vec{x}, \vec{y}) \\ &= \int_{x,y} [m + \mathcal{V}(\vec{x}, \vec{y})] f(\vec{x}, \vec{y}) \bar{\partial}J^a(\vec{x})U^{ab}(\vec{x}, \vec{y})\chi^b(\vec{y}) + \mathcal{O}(|\vec{x} - \vec{y}|) \end{aligned} \quad (43)$$

The normal ordering contribution from the term similar to II is actually zero, due to color contractions.

$$\begin{aligned} II' &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) \left[[p_s(\vec{v}), \bar{\partial}J^a(\vec{x})]U^{ab}(\vec{x}, \vec{y}) [\bar{p}_r(\vec{u}), \chi^b(\vec{y})] \right. \\ &\quad \left. + \bar{\partial}J^a(\vec{x})[p_s(\vec{v}), U^{ab}(\vec{x}, \vec{y})] [\bar{p}_r(\vec{u}), \chi^b(\vec{y})] \right] f(\vec{x}, \vec{y}) \end{aligned} \quad (44)$$

Using the commutation relations (17) we find that the first term in II' can be written as

$$\begin{aligned} II'_{(a)} &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) [p_s(\vec{v}), \bar{\partial}J^a(\vec{x})] U^{ab}(\vec{x}, \vec{y}) [\bar{p}_r(\vec{u}), \chi^b(\vec{y})] f(\vec{x}, \vec{y}) \\ &= m \int_{z,w} \bar{\partial}_z \Lambda_{ra}(\vec{w}, \vec{z}) [K(z, \bar{w})K^{-1}(w, \bar{w})]_{ab} f^{brs} \chi^s(\vec{w}) f(\vec{z}, \vec{w}) \end{aligned} \quad (45)$$

Explicit calculation of Λ_{ra} in [4] shows that (see eq.(A3) in Appendix)

$$\bar{\partial}_z \Lambda(\vec{w}, \vec{z}) = \frac{e^{-\alpha/2}}{2\pi\epsilon} \left[K(w, \bar{w})K^{-1}(z, \bar{w}) + \sum_{A,B \geq 1} (z-w)^A (\bar{z}-\bar{w})^B F_{AB}(\vec{w}) + \mathcal{O}(\epsilon) \right] \quad (46)$$

where $\alpha = |\vec{z} - \vec{w}|^2/\epsilon$. Inserting (46) into (45) we find that only the first term in (46) might contribute a finite ϵ^0 -order term, but this is actually zero due to color contraction.

$$\begin{aligned} II'_{(a)} &= m \int_{z,w} \frac{e^{-\alpha/2}}{2\epsilon\pi} [K(\vec{w})K^{-1}(z, \bar{w})]_{ra} [K(z, \bar{w})K^{-1}(\vec{w})]_{ab} f^{brs} \chi^s(\vec{w}) f(\vec{z}, \vec{w}) \\ &= 0 \end{aligned} \quad (47)$$

The second term in (44) can be similarly evaluated

$$\begin{aligned} II'_{(b)} &= \frac{e^2}{2} \int \Pi_{rs}(\vec{u}, \vec{v}) \bar{\partial}J^a(\vec{x}) [p_s(\vec{v}), U^{ab}(\vec{x}, \vec{y})] [\bar{p}_r(\vec{u}), \chi^b(\vec{y})] f(\vec{x}, \vec{y}) \\ &= -\frac{e^2}{2} \int \bar{\partial}J^a(\vec{x}) [\Pi_{rs}(\vec{y}; x, \bar{y}) - \Pi_{rs}(\vec{y}; \vec{y})] f^{rbc} f^{sml} K^{al}(x, \bar{y}) K^{bm}(y, \bar{y}) \chi^c(\vec{y}) f(\vec{x}, \vec{y}) \end{aligned} \quad (48)$$

In writing (48) we used the fact that

$$[p_s(\vec{v}), U^{ab}(\vec{x}, \vec{y})] = f^{sml} K^{al}(x, \bar{y}) K^{bm}(y, \bar{y}) [\delta(v-x) - \delta(v-y)] \delta(\bar{v}-\bar{y}) \quad (49)$$

Explicit calculation shows that

$$\Pi_{rs}(\vec{y}; x, \vec{y}) - \Pi_{rs}(\vec{y}; \vec{y}) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(x-y)^n}{nn!} \partial_y^n K_{rs}(\vec{y}) \quad (50)$$

Inserting (50) into (48) shows that the contribution of $II'_{(b)}$ is, indeed, negligible for very small separations. So we find that

$$(T\mathcal{O}_G) = \int_{x,y} [m + \mathcal{V}(\vec{x}, \vec{y})] f(\vec{x}, \vec{y}) \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y}) \chi^b(\vec{y}) + \mathcal{O}(|\vec{x} - \vec{y}|) \quad (51)$$

Finally, the remaining terms in (37) can be simplified using

$$[T, V_{YM}] = 2m V_{YM} + \frac{4\pi}{c_A} \int \mathcal{D} \bar{\partial} J \frac{\delta}{\delta J} \quad (52)$$

The term $2mV_{YM}$ is due to both \bar{p} and p acting on V_{YM} ; this is not needed for the terms under consideration. We then find

$$\begin{aligned} & -\frac{e^2}{4m} \int \Pi_{rs}(u, v) \{ [\bar{p}_r(u), V_{YM}] [p_s(v), \mathcal{O}_G] + [p_s(v), V_{YM}] [\bar{p}_r(u), \mathcal{O}_G] \} \\ & = \int_{x,y} \left[-\frac{\nabla_x^2}{2m} \right] f(\vec{x}, \vec{y}) : \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y}) \chi^b(\vec{y}) : + \mathcal{O}(|\vec{x} - \vec{y}|) \end{aligned} \quad (53)$$

The $\epsilon \rightarrow 0$ limit of (20), which has been used to simplify the right hand side of (52), is adequate for the calculation of the right hand side of (53). Notice that the resulting factor of $-\nabla^2/2m$, along with the mass m in (51), will reproduce the first two terms of the relativistic combination $\sqrt{m^2 - \nabla^2}$. The remainder is of order $|\vec{x} - \vec{y}|$.

Collecting various terms and results, we get

$$\begin{aligned} \mathcal{H} \Psi_G &= \int_{x,y} \left\{ \left[M + m - \frac{\nabla_x^2}{2m} + \mathcal{V}(\vec{x}, \vec{y}) \right] f(\vec{x}, \vec{y}) \right\} \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y}) \chi^b(\vec{y}) \Psi_0 \\ &\quad + \mathcal{O}(|\vec{x} - \vec{y}|) \end{aligned} \quad (54)$$

So far, we have not considered the addition of \mathcal{Q} as in (23). As mentioned earlier, the Coulomb potential in (51) is of the form $\text{Ein}(|\vec{x} - \vec{y}|^2/2\epsilon) \approx \log(|\vec{x} - \vec{y}|^2/2\epsilon)$. From the definition of \mathcal{Q} ,

$$\begin{aligned} \mathcal{Q} &\int \bar{\partial} J^a(\vec{x}) U^{ab}(\vec{x}, \vec{y}) \chi^b(\vec{y}) f(\vec{x}, \vec{y}) \\ &= \epsilon \int \sigma(\vec{u}, \vec{v}, \epsilon) K_{rs}(u, \vec{v}) [\bar{p}_r(\vec{u}), \bar{\partial} J^a(\vec{x})] [p_s(\vec{v}), U^{ab}(x, y)] \chi^b(\vec{y}) f(\vec{x}, \vec{y}) \\ &= \frac{c_A}{\pi} \int \bar{\partial} J^a(\vec{x}) U^{ab}(x, y) \chi^b(\vec{y}) f(\vec{x}, \vec{y}) \end{aligned} \quad (55)$$

where, in the first step, we have indicated the term which gives a nonzero result among the various terms generated by the action of \mathcal{Q} . We have evaluated the action of \mathcal{Q} on $\bar{\partial}J^a(\vec{x})U^{ab}(\vec{x},\vec{y})\bar{\partial}J^b(\vec{y})$ before [4]; the nonzero term has the same structure as in (55), except for the replacement of χ^b by $\bar{\partial}J^b$. Thus, the result in (55) is exactly of the same form as obtained for $\bar{\partial}J^a(\vec{x})U^{ab}(\vec{x},\vec{y})\bar{\partial}J^b(\vec{y})$.

The addition of $(e^2/2)\log(2\epsilon/\lambda)\mathcal{Q}$ to the definition of T thus leads to the result

$$\begin{aligned} \mathcal{H} \Psi_G = & \int_{x,y} \left\{ \left[M + m - \frac{\nabla_x^2}{2m} + \mathcal{V}(\vec{x},\vec{y},\lambda) \right] f(\vec{x},\vec{y}) \right\} \bar{\partial}J^a(\vec{x})U^{ab}(\vec{x},\vec{y})\chi^b(\vec{y}) \Psi_0 \\ & + \mathcal{O}(|\vec{x}-\vec{y}|) \end{aligned} \quad (56)$$

where $\mathcal{V}(\vec{x},\vec{y},\lambda) = m \left(\text{Ein}(|\vec{x}-\vec{y}|^2/2\epsilon) + \log(2\epsilon/\lambda) \right) \approx m \log(|\vec{x}-\vec{y}|^2/\lambda)$. (Notice that if $f(\vec{x},\vec{y})$ is of the form $\sigma(\vec{x},\vec{y},\lambda)$, the Coulomb potential vanishes.) Thus, within the approximations used for this continuum strong coupling expansion, we see that the state Ψ_G becomes an eigenstate of the Hamiltonian \mathcal{H} with eigenvalue E if $f(\vec{x},\vec{y})$ obeys the eigenvalue equation

$$\left[M + m - \frac{\nabla_x^2}{2m} + \mathcal{V}(\vec{x},\vec{y},\lambda) \right] f(\vec{x},\vec{y}) = E f(\vec{x},\vec{y}) \quad (57)$$

This equation shows that there is a constituent picture for the gluelump. The gauge-invariant gluon of mass m binds to the external charge via the Coulomb potential. Since we have arrived at this equation keeping only the F^2 -term in (Yang-Mills part of) $\log \Psi_0$, this is appropriate for spatial momenta small compared to m . The kinetic energy for the “gluon” is thus the nonrelativistic expression. As mentioned before, this is the beginning of a series which sums up to $\sqrt{m^2 - \nabla^2}$ [5]. Further, we find only the Coulomb potential. The key question is whether these approximations are adequate. If we are interested in a prediction in the continuum strong coupling limit, it is adequate to keep the nonrelativistic kinetic energy term. We cannot say, *a priori*, whether this is sufficient for comparison with lattice simulations, since a direct correspondence of the kinematic regimes in the two approaches is not obtained. However, experience with the comparison of the string tension and glueball masses suggests that this is a good starting point.

The question of the potential energy $\mathcal{V}(\vec{x},\vec{y})$ is another matter. From the action of T_{YM} on $\bar{\partial}J(\vec{x})U(\vec{x},\vec{y})\chi(\vec{y})$, we get the Coulomb limit of the potential. When effects due to the vacuum wave function are included, this potential must tend to the linear potential at large distances. We know this is so, from the existence of a nonzero string tension. However,

a direct calculation is difficult and we shall present a somewhat indirect argument in the next section. We will also use a slightly different definition of Ψ_G which is more convenient. On the lattice side, the potential energy is seen to rise linearly with distance, with a value of the slope or string tension which is in agreement with our calculations, up to a point, and then it flattens out, indicative of the breaking of the string. Thus, just at the point where the string breaks, the constituents of the singlet state (the external charge and the “gluon”) are separated far enough to be in the regime of the linear potential. For a sensible comparison with lattice data, we must understand and incorporate the linear potential. We take up this task in the next section.

However, already at this stage, it is worth noting that, if we consider the strong binding limit (when the separation $|\vec{x} - \vec{y}|$ is negligible) and also the strong coupling limit where $-\nabla^2 \ll m^2$, the energy in (57) can be approximated by $M + m$. The resulting value $2(E_0 - M) = 2m$ may be taken as a lower bound on the string-breaking energy.

5 The argument for the linear potential

The strategy we will consider for extracting the linear potential is as follows. We consider a state of the form

$$|\tilde{\alpha}\rangle = \int_{x,y} f(\vec{x}, \vec{y}) \psi^{\dagger a}(\vec{x}) W^{ab}(\vec{x}, \vec{y}) \phi^{\dagger b}(\vec{y}) |\alpha\rangle \quad (58)$$

where $W(\vec{x}, \vec{y})$ is the open Wilson line connecting points \vec{x} and \vec{y} , $W(\vec{x}, \vec{y}) = \mathcal{P} \exp \left(- \int_y^x A \right)$ and ψ and ϕ are heavy fields. $|\alpha\rangle$ specifies the state of the gauge field and can be expressed as $|\alpha\rangle = \mathcal{O}_\alpha |0\rangle$, where \mathcal{O}_α is constructed from the gauge fields only.

The matrix element of $e^{-\mathcal{H}\tau}$ for states of the form (58) can be related to the expectation value of the Wilson loop; the latter can be expressed in terms of the area law and the linear potential extracted from it. We use two dissimilar fields ψ and ϕ to avoid the possibility of mutual annihilation during the (Euclidean) time-evolution.

The action for the theory is taken as

$$S = \int d^3x \left[(\overline{D\psi})^a (D\psi)^a + (\overline{D\phi})^a (D\phi)^a - M^2 (\bar{\psi}^a \psi^a + \bar{\phi}^a \phi^a) \right] + S_{YM} \quad (59)$$

The Hamiltonian has the form

$$\mathcal{H} = \mathcal{H}_{YM} + \mathcal{H}_{scalar} + \mathcal{H}_{int} \quad (60)$$

where the last term involves interactions between ψ and A and between ϕ and A . This Hamiltonian commutes with N_ψ , the number of ψ -particles and N_ϕ , the number of ϕ -particles, separately. Thus matrix elements of the Hamiltonian are block-diagonal and we can restrict attention to the $N_\psi = N_\phi = 1$ block consistently. (A real adjoint scalar particle can, in principle, annihilate with another such particle. This would be an added complication which is irrelevant for our approach to the linear potential. So we avoid it by choosing two distinct complex scalars to construct the state. This is not the minimal way of doing this. A minimal way would involve one complex scalar, but the final results are the same, so we use the more convenient choice of two complex scalars.)

First consider the normalization of states of the form (58). For this, the states are at equal time. We can use $\psi(\vec{x})\psi^\dagger(\vec{y})|\alpha\rangle = \delta(\vec{x} - \vec{y})|\alpha\rangle$, since \mathcal{O}_α has no ψ 's in it. We then find

$$\langle\tilde{\alpha}|\tilde{\alpha}\rangle = \int_{x,y} f^*(\vec{x}, \vec{y}) f(\vec{x}, \vec{y}) \langle\alpha|\alpha\rangle \quad (61)$$

The evaluation of $\langle\alpha|\alpha\rangle$ can also be simplified if we approximate the vacuum wave function by the expression (36). We then find

$$\langle\alpha|\alpha\rangle = \int d\mu(\mathcal{A}/\mathcal{G}_*) \exp\left[-\frac{1}{4g^2} \int d^2x F^2\right] \bar{\mathcal{O}}_\alpha \mathcal{O}_\alpha \quad (62)$$

We can choose the gauge $A_1 = 0$, whereupon the action is quadratic in A_2 with a propagator

$$\langle A_2^a(\vec{x}) A_2^b(\vec{y}) \rangle = \delta^{ab} \delta(x_1 - y_1) \int \frac{dk_2}{2\pi} e^{ik_2(x_2 - y_2)} \frac{1}{k_2^2} \quad (63)$$

The actual evaluation of (62) can then be done by Wick contractions using (63).

We now turn to the matrix element $\langle\tilde{\beta}|e^{-\mathcal{H}\tau}|\tilde{\alpha}\rangle$, where $|\tilde{\beta}\rangle$ is another state similar to $|\tilde{\alpha}\rangle$, with \mathcal{O}_β and a function $h(\vec{x}, \vec{y})$ replacing \mathcal{O}_α and $f(\vec{x}, \vec{y})$, respectively.

$$\begin{aligned} \langle\tilde{\beta}|e^{-\mathcal{H}\tau}|\tilde{\alpha}\rangle &= \int h^*(\vec{x}', \vec{y}') f(\vec{x}, \vec{y}) \langle\beta|\phi(\vec{y}') W(\vec{y}', \vec{x}') \psi(\vec{x}') e^{-\mathcal{H}\tau} \psi^\dagger(\vec{x}) W(\vec{x}, \vec{y}) \phi^\dagger(\vec{y})|\alpha\rangle \\ &= \int h^*(\vec{x}', \vec{y}') f(\vec{x}, \vec{y}) \langle 0| [\bar{\mathcal{O}}_\beta \phi(\vec{y}') W(\vec{y}', \vec{x}') \psi(\vec{x}')]_\tau \psi^\dagger(\vec{x}) W(\vec{x}, \vec{y}) \phi^\dagger(\vec{y}) \mathcal{O}_\alpha |0\rangle \end{aligned} \quad (64)$$

The set of fields at the left end have been shifted by τ in Euclidean time. Since the fields are heavy, we can write, for large M , $\phi(\tau) = e^{\mathcal{H}\tau} \phi e^{-\mathcal{H}\tau} \approx e^{-M\tau} \phi(0)$, so that $\langle\phi(\tau, \vec{x}') \phi^\dagger(0, \vec{x})\rangle \approx e^{-M\tau} \delta^{(2)}(\vec{x} - \vec{x}')$. We can then simplify the matrix element as

$$\langle\tilde{\beta}|e^{-\mathcal{H}\tau}|\tilde{\alpha}\rangle = \int h^*(\vec{x}, \vec{y}) f(\vec{x}, \vec{y}) e^{-2M\tau} \langle 0| \bar{\mathcal{O}}_\beta(\tau) W(C) \mathcal{O}_\alpha(0) |0\rangle \quad (65)$$

where $\mathcal{O}_\alpha(0) = \mathcal{O}_\alpha(\tau = 0)$ and $W(C)$ is the Wilson loop $W(C) = \text{Tr} \mathcal{P} \exp(-\oint_C A)$. The path C is a rectangle connecting the points $(\vec{x}, 0)$, $(\vec{y}, 0)$, (\vec{y}, τ) and (\vec{x}, τ) . The evaluation of the remaining expectation value in (65) can be understood using Wick contractions. There are two types of contractions involved:

- 1) Wick contractions within $W(C)$ and separately within $\bar{\mathcal{O}}_\beta \mathcal{O}_\alpha$
- 2) Wick contractions between $W(C)$ and $\bar{\mathcal{O}}_\beta \mathcal{O}_\alpha$.

We neglect the second type of contractions; we comment on this later. In this approximation, we have

$$\begin{aligned}
\langle 0 | [\bar{\mathcal{O}}_\beta]_\tau W(C) \mathcal{O}_\alpha(0) | 0 \rangle &\approx \langle 0 | \bar{\mathcal{O}}_\beta(\tau) \mathcal{O}_\alpha(0) | 0 \rangle \langle 0 | W(C) | 0 \rangle \\
&= e^{-\sigma |\vec{x} - \vec{y}| \tau} \langle 0 | \bar{\mathcal{O}}_\beta(\tau) \mathcal{O}_\alpha(0) | 0 \rangle \\
&= e^{-\sigma |\vec{x} - \vec{y}| \tau} \langle \beta | e^{-\mathcal{H}_{YM} \tau} | \alpha \rangle
\end{aligned} \tag{66}$$

Once we approximate as in the first line of this equation, the expectation value of $W(C)$ can be evaluated, by Euclidean invariance, as an equal-time average. Also, notice that only $e^{-\mathcal{H}_{YM} \tau}$ appears in the last line. Thus, within the approximations we have made,

$$\langle \beta | \int h^* \phi W \psi e^{-\mathcal{H} \tau} \int f \psi^\dagger W \phi^\dagger | \alpha \rangle = \int h^* f e^{-2M\tau - \sigma |\vec{x} - \vec{y}| \tau} \langle \beta | e^{-\mathcal{H}_{YM} \tau} | \alpha \rangle \tag{67}$$

We may rewrite part of this integral as

$$\int_{x,y} h^*(\vec{x}, \vec{y}) f(\vec{x}, \vec{y}) e^{-\sigma |\vec{x} - \vec{y}| \tau} = \langle 0 | \left[\int h^* \phi W \psi \right]_{\tau=0} e^{-\sigma |\vec{x} - \vec{y}| \tau} \left[\int f \psi^\dagger W \phi^\dagger \right]_{\tau=0} | 0 \rangle \tag{68}$$

Using this result, we finally get

$$\langle \beta | \int h^* \phi W \psi e^{-\mathcal{H} \tau} \int f \psi^\dagger W \phi^\dagger | \alpha \rangle = \langle \tilde{\beta} | e^{-(2M + \sigma |\vec{x} - \vec{y}|) \tau} \int f \psi^\dagger W \phi^\dagger e^{-\mathcal{H}_{YM} \tau} | \alpha \rangle \tag{69}$$

Since $|\tilde{\beta}\rangle$ is completely arbitrary within the $N_\psi = N_\phi = 1$ block, we conclude, by taking terms linear in τ ,

$$\mathcal{H} \int f \psi^\dagger W \phi^\dagger | \alpha \rangle = (2M + \sigma |\vec{x} - \vec{y}|) \int f \psi^\dagger W \phi^\dagger | \alpha \rangle + \int f \psi^\dagger W \phi^\dagger \mathcal{H}_{YM} \mathcal{O}_\alpha | 0 \rangle \tag{70}$$

Some comments on the approximations used in this calculation are in order at this point.

1. We have approximated the vacuum wave function by equation (36). This is adequate for the linear potential, but there can be corrections due to the higher terms (in powers of momenta and powers of J) in the exponent. We hope to address the issue of higher order terms elsewhere.

2. Wick contractions between $W(C)$ and $\bar{\mathcal{O}}_\beta \mathcal{O}_\alpha$ have been neglected. These can lead to structures different from $W(C)$ on the right hand side and hence to nondiagonal pieces for the first term in (70). These are small if the linear potential dominates. The lattice data suggest that this is the case, as explained earlier. Further, since $W(C)$ has no color charges, these contractions will be suppressed by powers of $1/N$.
3. We need large M limit to avoid virtual and real processes with additional numbers of ψ and ϕ particles. Such processes could also modify the Yang-Mills vacuum, if we do not take the large M limit. (This is a quenched approximation for these particles.)

Consider now the case of the gluelump. To get the wave function of the gluelump, we can take equation (70) with $\mathcal{O}_\alpha = 1$ and apply on this equation the operator $\int \bar{\partial} J^a \eta^a$, where $\eta^a = M^{\dagger ak} \psi^k$. This has the effect of replacing ψ^\dagger by $\bar{\partial} J^a M^{\dagger ak}$. Further the action of ϕ^\dagger on the vacuum wave function for the heavy scalars gives $\phi = M^{\dagger -1} \chi$. Thus we find

$$\begin{aligned} \int \bar{\partial} J \eta \psi^\dagger W \phi^\dagger \Psi_0 &= \bar{\partial} J^a(\vec{x}) \left(M^{\dagger ak}(\vec{x}) W^{kl}(\vec{x}, \vec{y}) M^{\dagger -1 lb}(\vec{y}) \right) \chi^b(\vec{y}) \Psi_0 \\ &\equiv \bar{\partial} J^a(\vec{x}) \widetilde{W}^{ab}(\vec{x}, \vec{y}) \chi^b(\vec{y}) \Psi_0 \end{aligned} \quad (71)$$

The operator $\widetilde{W}^{ab}(\vec{x}, \vec{y})$ may be written as

$$\widetilde{W}^{ab}(\vec{x}, \vec{y}) = \left[\mathcal{P} \exp \left(\int_y^x \partial K K^{-1} \right) \right]^{ab} \quad (72)$$

It is an open Wilson line for the holomorphic component of the connection $\partial K K^{-1}$. (The result (72) follows from the fact that we may think of (A, \bar{A}) as a complex gauge transform (by the matrix M^\dagger) of $(-\partial H H^{-1}, 0)$, i.e., $(A, \bar{A}) = (-\partial H H^{-1}, 0)^{M^\dagger}$, and because $W^{kl}(\vec{x}, \vec{y}, A^g) = g^{-1}(\vec{x}) W^{kl}(\vec{x}, \vec{y}) g(\vec{y})$ for $A^g = g^{-1} A g + g^{-1} \partial g$.)

As in the case of W , \widetilde{W} depends on the curve chosen from \vec{y} to \vec{x} . For a short separation $|\vec{x} - \vec{y}|$, we can write

$$\widetilde{W}(\vec{x}, \vec{y}) = K(x, \vec{y}) K^{-1}(y, \vec{y}) = U(\vec{x}, \vec{y}) \quad (73)$$

so that the functional $\bar{\partial} J^a \widetilde{W}^{ab} \chi^b$ agrees with the gluelump state used in section 4. For a longer separation, there are differences. Using the composition property of path-ordered exponentials, we find, for a straight-line path,

$$\begin{aligned} \widetilde{W}(y + 2\epsilon, y) &= K(y + 2\epsilon, \vec{y} + \vec{\epsilon}) K^{-1}(y + \epsilon, \vec{y} + \vec{\epsilon}) K(y + \epsilon, \vec{y}) K^{-1}(y, \vec{y}) \\ &\approx K(y + 2\epsilon, \vec{y}) K^{-1}(y, \vec{y}) + \epsilon \vec{\epsilon} \bar{\partial}(\partial K K^{-1}) + \dots \\ &= U(y + 2\epsilon, \vec{y}) + \epsilon \vec{\epsilon} \bar{\partial}(\partial K K^{-1}) + \dots \end{aligned} \quad (74)$$

Thus the Wilson line \widetilde{W} will differ from $U(\vec{x}, \vec{y}) = K(x, \vec{y})K^{-1}(y, \vec{y})$ by terms involving $\bar{\partial}(\partial K K^{-1})$ and higher derivatives of it when the series is continued. Such terms evidently correspond to deformations of the path.

The gluelump state can be defined, at finite separations, by either $\bar{\partial}J^a(\vec{x})U^{ab}(\vec{x}, \vec{y})\chi^b(\vec{y})$ or $\bar{\partial}J^a(\vec{x})\widetilde{W}^{ab}(\vec{x}, \vec{y})\chi^b(\vec{y})$. Which is more advantageous will depend on which leads to a simpler equation within the approximations we are using. The emergence of equation (70) suggests that $\bar{\partial}J^a(\vec{x})\widetilde{W}^{ab}(\vec{x}, \vec{y})\chi^b(\vec{y})$ is a simpler functional to use, and so we redefine the gluelump as

$$\Psi_G = \int d^2x d^2y f(\vec{x}, \vec{y}) \bar{\partial}J^a(\vec{x}) \widetilde{W}^{ab}(\vec{x}, \vec{y}) \chi^b(\vec{y}) \Psi_0 \quad (75)$$

Using (71), we then find

$$\mathcal{H} \int f \bar{\partial}J \widetilde{W} \chi \Psi_0 = [\mathcal{H}, \int \bar{\partial}J \eta] \int f \psi^\dagger W \phi^\dagger \Psi_0 + (2M + \sigma|\vec{x} - \vec{y}|) \int f \bar{\partial}J \widetilde{W} \chi \Psi_0 \quad (76)$$

The operator $\bar{\partial}J\eta$ is a local operator. This must be defined, in a regularized way, by an expression of the form \mathcal{O}_G where $f(\vec{x}, \vec{y})$ is replaced by $\sigma(\vec{x}, \vec{y}, \lambda)$; then we must take the small λ limit. In this case, we can use the result (56); the Coulomb potential is zero and we can write

$$[\mathcal{H}, \int \bar{\partial}J \eta] \Psi_0 = \int \sigma(\vec{u}, \vec{v}, \lambda) \left[\left(-M + m - \frac{\nabla_u^2}{2m} \right) \bar{\partial}J^a(\vec{u}) \right] U^{ab}(\vec{u}, \vec{v}) \eta^b(\vec{v}) \Psi_0 + \dots \quad (77)$$

where $U^{ab}(\vec{u}, \vec{v})$ is used because of the point-splitting needed to define $\bar{\partial}J\eta$. The omitted terms in (77) correspond to terms with either \bar{p} or p acting on $\int \bar{\partial}J\eta$ and the other left free so that it can act on $\int f \psi^\dagger W \phi^\dagger$ in (76); i.e., they are of the form

$$\begin{aligned} & \frac{e^2}{2} \int \Pi_{rs}(u, v) \left\{ [\bar{p}_r(u), \int \bar{\partial}J \eta] f \psi^\dagger [p_s(v), W] \phi^\dagger + [p_s(v), \int \bar{\partial}J \eta] f \psi^\dagger [\bar{p}_r(u), W] \phi^\dagger \right\} \Psi_0 \\ &= \frac{e^2}{2} \int \Pi_{rs}(u, v) \left\{ [\bar{p}_r(u), \int \bar{\partial}J \eta] f \eta^\dagger [p_s(v), \widetilde{W}] \chi^\dagger + [p_s(v), \int \bar{\partial}J \eta] f \psi^\dagger [\bar{p}_r(u), W] \phi^\dagger \right\} \Psi_0 \end{aligned} \quad (78)$$

One of the terms on the right hand side, namely, $e^2/2 \int \Pi_{rs}(\vec{u}, \vec{v}) [\bar{p}_r(\vec{u}), \int \bar{\partial}J^a][p_s(\vec{v}), \widetilde{W}^{ab}] \chi^{\dagger b}$ is the type of term which led to the Coulomb potential in the calculation of section 4. So one might wonder whether this can generate a Coulomb term in addition to the linear potential. This is not the case. In fact, the Coulomb potential is due to the term $f_{arm} \bar{\partial}J^m \delta(\vec{x} - \vec{u})$ in the commutator $[\bar{p}_r(\vec{u}), \bar{\partial}J^a(\vec{x})] = -i \bar{\partial} \mathcal{D}_{ar} \delta(\vec{x} - \vec{u})$. Notice that in (78) we have the combination $[\bar{p}_r(\vec{u}), \int \bar{\partial}J \eta]$ and, since $[\bar{p}_r(\vec{u}), \eta^a(\vec{x})] = f_{arm} \eta^m \delta(\vec{x} - \vec{u})$ as well, the relevant term is

actually zero. (This too has to be understood with point-splitting, but this particular result is basically the same as in the unregulated calculation.) We will neglect the remaining terms in (78), which lead to new operator structures. These terms lead to operator structures with more powers of J . The gluelump state we consider can have overlap with states which have the same quantum numbers as $\bar{\partial}J\widetilde{W}\chi$. The fact that the Hamiltonian acting on $\bar{\partial}J\widetilde{W}\chi$ can evolve it into other structures is indicative of this possibility of mixing. However, as is well known in quantum mechanics, the effect of such nondiagonal terms in \mathcal{H} comes with energy denominators and can be taken to be small if the differences between the energies of the lowest gluelump state (which is what we are interested in) and higher states are large enough. We expect this to be the case at large enough coupling, since the differences must go like m . But ultimately it is to be justified *a posteriori*.³

With this understanding of various terms, combining the results in (76) and (77), we finally have, as λ in (77) is taken to be very small,

$$\mathcal{H} \Psi_G = \int \left[M + m - \frac{\nabla_x^2}{2m} + \sigma|\vec{x} - \vec{y}| \right] f(\vec{x}, \vec{y}) \bar{\partial}J^a(\vec{x})\widetilde{W}^{ab}(\vec{x}, \vec{y})\chi^b(\vec{y})\Psi_0 + \dots \quad (79)$$

The approximations involved in this equation are given by the comments after equation (70) and by the comments about the omitted terms in (77, 78).

Equation (79) shows that the eigenvalue equation we need for the gluelump states is

$$\left[M + m - \frac{\nabla_x^2}{2m} + \sigma|\vec{x} - \vec{y}| \right] f(\vec{x}, \vec{y}) = E f(\vec{x}, \vec{y}) \quad (80)$$

By our general arguments, the string-breaking energy is given by $V_* = 2(E_0 - M)$ where E_0 is the ground state energy eigenvalue of the equation (80). We now turn to the determination of this value.

6 A variational estimate for string breaking

Since we are interested in the ground state eigenvalue, the simplest estimate is given by a variational calculation. Removing the center of mass motion (which is zero as $M \rightarrow \infty$) the

³The transition amplitude from the unbroken string state to the gluelumps is seen to be very small from lattice data, even for $SU(2)$ [12, 13]. This is another indication that the off-diagonal elements mentioned above might be small.

energy functional corresponding to (80) is

$$\begin{aligned} E &= M + m + \frac{1}{\mathcal{N}} \int d^2x \left[\frac{|\nabla f|^2}{2m} + \sigma |\vec{x}| |f|^2 \right] \\ \mathcal{N} &= \int d^2x |f|^2 \end{aligned} \quad (81)$$

We take a variational ansatz of the form $f = \exp(-\beta|x|^\mu)$, where both β and μ are regarded as variational parameters. The minimum of $E(\beta, \mu)$ with respect to β is found to be at $\beta = \beta_*$,

$$\beta_* = \left(\frac{2^{(\mu-3)/\mu} (2m\sigma) \Gamma(3/\mu)}{\mu^2} \right)^{\mu/3} \quad (82)$$

A formula for the energy eigenvalue $E(\beta_*, \mu)$ as a function of μ can now be obtained.

$$\begin{aligned} E(\beta_*, \mu) &= M + m + \left[\frac{2^{-(2\mu+1)/\mu}}{2m\Gamma(2/\mu)} \left(\frac{2^{(\mu-3)/\mu} 2m\sigma \Gamma(3/\mu)}{\mu^2} \right)^{-1/3} \right. \\ &\quad \left. \times \left(8m\sigma \Gamma(3/\mu) + 8^{1/\mu} \mu^2 \left(\frac{2^{(\mu-3)/\mu} 2m\sigma \Gamma(3/\mu)}{\mu^2} \right) \right) \right] \end{aligned} \quad (83)$$

The string tension for a representation R is given by (2); for the present calculation, we need the adjoint string tension, $\sigma_A = \pi m^2$. Using this value of the string tension, the formula for $E(\beta_*, \mu)$ can be minimized numerically. The minimum is obtained at $\mu_* = 1.752$. This gives $E_0 - M = E(\beta_*, \mu_*) - M = 3.958m$, corresponding to a string-breaking energy of $V_{*cal} = 2(E(\beta_*, \mu_*) - M) = 7.916 m$.

To check the robustness of the variational result, we have also tried to estimate the string breaking energy using a more general variational ansatz of the form

$$f = \frac{e^{-\beta|x|^\mu}}{(1 + |x|)^\nu} \quad (84)$$

Such an ansatz is suggested by the large x solution of the two-dimensional Schrödinger equation with a linear potential, whose solutions fall off, at large $|x|$, as

$$\frac{\text{Ai}((2m\sigma)^{1/3}|x|)}{\sqrt{|x|}} \approx \frac{\exp\left[-\frac{2}{3}((2m\sigma)^{1/3}|x|)^{3/2}\right]}{|x|^{3/4}} \quad (85)$$

where Ai is the Airy function. However, a numerical minimization of $E(\beta, \mu, \nu)$, with respect to the three variational parameters β, μ, ν , shows that the lowest estimate for E is provided by $\nu = 0$, i.e., by the exponential ansatz reported above.

We are now in a position to compare this with lattice estimates. The latest data on this is for $SU(2)$, given by Kratochvila and Forcrand [13]. The string-breaking energy is given by $V_{*lat} \approx 8.68 \, m$. This arises as follows. V_{*lat} is computed in terms of the lattice spacing a as $V_{*lat} \approx 2.063a^{-1}$. In the same set-up, the fundamental string tension is given by $\sigma_F = 0.0625a^{-2}$. This can be used to write $V_{*lat} \approx 2.06 \, m \sqrt{\pi c_F / 0.0625 c_A} \approx 8.68 \, m$. For our analytic calculation, we then have $|V_{*lat} - V_{*cal}|/V_{*lat} \approx 8.76\%$.

It is interesting also to make a similar estimate for the 0^{++} glueball mass, at least as a further check on the approximations used. If we use our picture of two constituents of mass m each, interacting via the linear potential, the eigenvalue equation is

$$\left[2m - \frac{\nabla_x^2}{2m} - \frac{\nabla_y^2}{2m} + \sigma |\vec{x} - \vec{y}| \right] f(\vec{x}, \vec{y}) = E f(\vec{x}, \vec{y}) \quad (86)$$

The only difference, compared to what we have done for the gluelump, is that the reduced mass should now be $\frac{1}{2}m$. This leads to

$$M_{0^{++}} = 5.73 \, m \quad (87)$$

The difference with the lattice estimate of $5.17 \, m$ [9] is 10.83%. In this case, of course, since both constituents have mass $m \ll M$, our $1/m$ -expansion and the nonrelativistic approximation are less reliable.

7 Discussion

A Hamiltonian approach to Yang-Mills theory in 2+1 dimensions was introduced several years ago by two of the authors [3, 4, 5]. In this paper we have extended the formalism to include matter fields, specifically scalar fields in the adjoint representation. First of all, this may be viewed as a first step towards analyzing supersymmetric theories. Secondly, it may be used for analyzing screening and string breaking. On general group-theoretic grounds we should expect screening of Wilson loops in the adjoint and other representations of zero N -ality. When the length of a string in one of the screenable representations is increased, it breaks at some point with the formation of two color singlet states, each of which is referred to as a gluelump. We obtained a Schrödinger equation for a gluelump state, the variational solution of which then gave an analytic estimate of the string-breaking energy. This is within 8.8% of the latest lattice estimates. The emerging picture is that of a bound

state between a constituent “gluon” $\bar{\partial}J$, of mass m , and a heavy adjoint scalar of mass M , interacting via a linear potential $V \sim \sigma_A r$. Some approximations are necessary for obtaining the Schrödinger equation, since most of the states under discussion are unstable; the approximations we have used are given in section 5. It should also be emphasized that we are calculating the energy at which the string breaks, the transition amplitude for the breaking is not calculated. (These are two different questions. The transition amplitude, as is well known, will be suppressed at large N .)

Specifically for the $(2+1)$ -dimensional theory, the string-breaking energy has not been analyzed extensively on the lattice except for the group $SU(2)$. We consider this to be an important question for which more lattice data are needed, for higher groups, and, in particular, the N -dependence.

Also, as explained in the text, string breaking is closely related to the issue of corrections to the formula (2) for the string tension. This matter will be taken up in a future publication.

VPN thanks David Gross for discussions at a preliminary stage of this work during a visit to KITP, Santa Barbara in 2004. We also acknowledge useful discussions with Barak Bringoltz. AA also wishes to thank the Max Planck Institut für Gravitationsphysik, where part of this work was carried out, for hospitality. This research was supported in part by the National Science Foundation grants PHY-0457304 and PHY-0555620 and by PSC-CUNY grants.

APPENDIX

Once the regulated Green’s functions (13) have been introduced, it is straightforward to evaluate the regularized form of the Hamiltonian and its action on various functionals. However, it is important to understand and follow the correct order of limits. Here we shall comment briefly on this issue.

We start by recalling some elementary properties of a covariant integral $I(p, \epsilon)$ resulting from some term in the perturbation expansion of a field theory. Here p denotes external momenta and ϵ is the short-distance cut-off. Once the integration over the loop momenta

are done, $I(p, \epsilon)$ is of the form

$$I(p, \epsilon) = \sum_{\alpha} \epsilon^{-\alpha} \mathcal{O}_{\alpha}(p) + \tilde{\mathcal{O}}(p) + \sum_{\beta} \epsilon^{\beta} \mathcal{O}'_{\beta} \quad (\text{A1})$$

The first set of terms involves negative powers of ϵ and represent potentially divergent terms as ϵ becomes very small. (We include $\alpha = 0$, which may be taken as a logarithmic divergence.) $\tilde{\mathcal{O}}(p)$ represents finite terms and the last set of terms vanish as $\epsilon \rightarrow 0$. At finite, but small, ϵ , the last set of terms are suppressed by powers of $p^2\epsilon$; they are negligible for small external momenta. While the divergent and finite terms are independent of the regulator (apart from the usual ambiguity of the subtraction points), the \mathcal{O}'_{β} terms are sensitive to the regulator. Of course, as $\epsilon \rightarrow 0$, this does not matter, but notice that for external momenta of the order of $\epsilon^{-1/2}$, these terms can contribute and constitute regularization ambiguities. The correct procedure is obviously that the result (A1) is to be used only for processes with external momenta small compared to $\epsilon^{-1/2}$.

A simple case which highlights this point is the following. A regulator does not have to respect symmetries, say, Lorentz invariance; the standard lattice regularization is an example. In a Lorentz-invariant theory, this can lead to Lorentz-violating terms of order $p\sqrt{\epsilon}$. If we consistently restrict our calculation to processes with momenta $p \ll \epsilon^{-1/2}$, these spurious regulator-dependent terms will be negligible.

In a Hamiltonian formulation, we define the regulated kinetic energy operator $T(\epsilon)$, given by (12), (13). Since it is a functional differential operator, to interpret it properly, we have to consider its action on different functionals. Consider a functional $\Psi(\lambda')$ with the average separation of points between fields in the functional being $\sqrt{\lambda'}$. When we act on this with $T(\epsilon)$, there are three types of terms possible.

1. Terms which diverge as $\epsilon \rightarrow 0$. If there are such terms, we must introduce a subtraction procedure to make $T(\epsilon)$ well-defined and eliminate these terms. (The redefinition of $T(\epsilon)$ with the addition of \mathcal{Q} in (23) may be viewed as an example of this.)
2. Terms which are finite as $\epsilon \rightarrow 0$. These are physical effects.
3. Terms which are of order ϵ . These can be neglected. However, if we take $\lambda' \sim \epsilon$, or $\lambda' \rightarrow 0$ at finite ϵ , these terms can give finite results, just as the \mathcal{O}'_{β} terms in (A1). Exactly as in the previous case, the correct procedure is to realize that the

$T(\epsilon)$ can only be used for functionals with $\lambda' \gg \epsilon$; otherwise there will be spurious regulator-dependent terms.

Keeping this in mind, all the calculations have to be done with $\epsilon \rightarrow 0$ first. A natural question is then: How do we define a “local” operator like $V_{YM} \sim \int B^2$? Note that this is needed for the Hamiltonian, and further we will need the commutator of T with this operator. From what was said before, V_{YM} must be defined in a regulated way with an average point-separation of $\sqrt{\lambda}$, with $\lambda \gg \epsilon$. Then the action of $T(\epsilon)$ on this is correctly obtained. To make the operator local, we then take $\lambda \rightarrow 0$. In other words, we must take ϵ, λ to zero, keeping $\lambda \gg \epsilon$. This is the correct procedure and it will yield unambiguous results. This procedure is even more crucial in a Hamiltonian framework; otherwise, spurious terms may appear and it may not even be obvious that they are spurious, since Lorentz symmetry is not manifest.

This procedure was used in [4, 5]. For the purposes of this paper, we note that an expression for Λ_{ra} was obtained in [4] as

$$\Lambda_{ra}(\vec{w}, \vec{z}) = \frac{1}{\pi(z-w)} [\delta_{ra} - (K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})K^{-1}(z, \bar{u}))_{ra} e^{-\alpha/2}] + \widetilde{\Lambda}_{ra}(\vec{w}, \vec{z}) \quad (\text{A2})$$

$$\begin{aligned} \widetilde{\Lambda}_{ra}(\vec{w}, \vec{z}) = & \frac{\epsilon}{\pi} e^{-\alpha/2} \left[-2\partial_z \frac{1}{z-w} \bar{\partial}_z (K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})K^{-1}(z, \bar{u})) \right. \\ & + \frac{2}{z-w} \bar{\partial}_z (K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})\partial_z K^{-1}(z, \bar{u})) \\ & + \left. \frac{2}{(z-w)(\bar{z}-\bar{w})} K(\vec{w})K^{-1}(u, \bar{w})K(\vec{u})\partial_z (K^{-1}(z, \bar{u})K(\vec{z}))K^{-1}(\vec{z}) \right]_{ra} \\ & + \mathcal{O}(\epsilon^2) \end{aligned} \quad (\text{A3})$$

where $u = \frac{1}{2}(z+w)$, $\bar{u} = \frac{1}{2}(\bar{z}+\bar{w})$ and $\alpha = (z-w)(\bar{z}-\bar{w})/\epsilon$. If we take $\sqrt{\epsilon}$ to be very small compared to the average separation of fields in functionals on which T acts, $\widetilde{\Lambda}_{ra}(\vec{w}, \vec{z})$ does not contribute and we can use the simplified form for $\Lambda_{ra}(\vec{w}, \vec{z})$, namely, expression (A2) with $\widetilde{\Lambda}_{ra}(\vec{w}, \vec{z})$ set to zero. In fact we can simplify the first term even further, approximating $\Lambda_{ra}(\vec{w}, \vec{z})$ as

$$\begin{aligned} \Lambda_{ra}(\vec{w}, \vec{z}) = & \frac{1}{\pi(z-w)} [\delta_{ra} - (K(\vec{w})K^{-1}(z, \bar{w}))_{ra} e^{-\alpha/2}] \\ & + \text{(terms of higher order in } \epsilon \text{ or } (z-w), (\bar{z}-\bar{w})) \end{aligned} \quad (\text{A4})$$

This is correct and adequate, but it must be kept in mind that T so defined acts on operators with a point-splitting separation $\sqrt{\lambda} \gg \sqrt{\epsilon}$, and in taking the small λ -limit we must preserve this inequality.

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